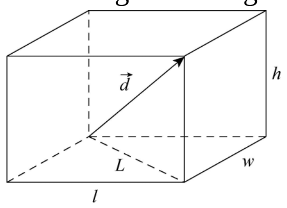
Chapter 3 *Vectors*

螞蟻在沙漠平原上無導引線索下如何知道回家的路？

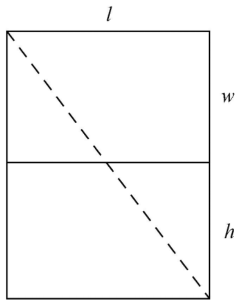
07. The length unit meter is understood throughout the calculation. (**a**) We compute the distance from one corner to the diametrically opposite corner:

$$d = (3.00^2+3.70^2+4.30^2)^{1/2} = 6.42 .$$

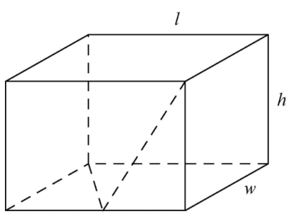
(**b**) The displacement vector is along the straight line from the beginning to the end point of the trip. Since a straight line is the shortest distance between two points, the length of the path cannot be less than the magnitude of the displacement.



(**c**) It can be greater, however. The fly might, for example, crawl along the edges of the room. Its displacement would be the same but the path length would be $\ell+w+h$ = 11.0 m. (**d**) The path length is the same as the magnitude of the displacement if the fly flies along the displacement vector. (**e**) We take the x axis to be out of the page, the y axis to be to the right, and the z axis to be upward.



Then the x component of the displacement is w = 3.70, the y component of the displacement is 4.30, and the z component is 3.00. Thus $\vec{d} = 3.70\,\hat{\mathrm{i}} + 4.30\,\hat{\mathrm{j}} + 3.00\,\hat{\mathrm{k}}$. An equally correct answer is obtained by interchanging the length, width, and height. (**f**) Suppose the path of the fly is as shown by the dotted lines on the upper diagram. Pretend there is a hinge where the front wall of the room joins the floor and lay the wall down as shown on the lower diagram. The shortest walking distance between the lower left back of the room and the upper right front corner is the dotted straight line shown on the diagram. Its length is



$$L_{min} = \sqrt{(w+h)^2+\ell^2}$$

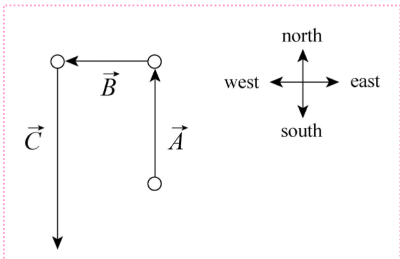
$$= \sqrt{(3.70+3.00)^2+4.30^2} = 7.96 \text{ (m)}.$$

08. We label the displacement vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ (and denote the result of their vector sum as $\vec{r}$). We choose *east* as the $\hat{\mathrm{i}}$ direction (+x direction) and *north* as the $\hat{\mathrm{j}}$ direction (+y direction). All distances are understood to be in kilometers. (**a**) The vector diagram representing the motion is shown below. (**b**) The final point is represented by

$$\vec{r} = \vec{A} + \vec{B} + \vec{C} = -2.4\,\hat{\mathrm{i}} - 2.1\,\hat{\mathrm{j}}.$$

whose magnitude is

$$|\vec{r}| = \sqrt{(-2.4)^2+(-2.1)^2} \approx 3.2 \text{ (km)}.$$



$$\vec{A} = 3.1\,\hat{\mathrm{j}}$$

$$\vec{B} = -2.4\,\hat{\mathrm{i}}$$

$$\vec{C} = -5.2\,\hat{\mathrm{j}}$$

(**c**) There are two possibilities for the angle:

$$\tan^{-1}\left(\frac{-2.1}{-2.4}\right) = 41°, \text{ or } 221°.$$

We choose the latter possibility since $\vec{r}$ is in the third quadrant. It should be noted that many graphical calculators have polar ↔ rectangular "shortcuts" that automatically produce the correct answer for angle (measured counterclockwise from the +x axis). We may phrase the angle, then, as 221° counterclockwise from East (a phrasing that sounds peculiar, at best) or as 41° south from west or 49° west from south. The resultant $\vec{r}$ is not shown in our sketch; it would be an arrow directed from the "tail" of $\vec{A}$ to the "head" of $\vec{C}$.

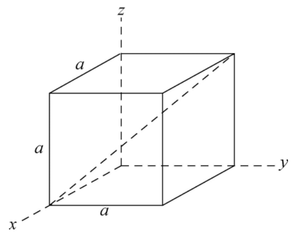
15. All distances in this solution are understood to be in meters. (**a**) $\vec{a}+\vec{b}$ = [4.0+ (–1.0)] $\hat{\mathrm{i}}$ +[(–3.0) + 1.0] $\hat{\mathrm{j}}$ + (1.0+4.0) $\hat{\mathrm{k}}$ = $3.0\,\hat{\mathrm{i}} - 2.0\,\hat{\mathrm{j}} + 5.0\,\hat{\mathrm{k}}$.

(**b**) $\vec{a}-\vec{b}$ = [4.0–(–1.0)] $\hat{\mathrm{i}}$ +[(–3.0)–1.0] $\hat{\mathrm{j}}$ + (1.0–4.0) $\hat{\mathrm{k}}$ = $5.0\,\hat{\mathrm{i}} - 4.0\,\hat{\mathrm{j}} - 3.0\,\hat{\mathrm{k}}$.

(**c**) The requirement $\vec{a}-\vec{b}+\vec{c} = 0$ leads to $\vec{c} = \vec{b}-\vec{a}$, which we note is the opposite of what we found in part (b). Thus, $\vec{c} = -5.0\,\hat{\mathrm{i}} + 4.0\,\hat{\mathrm{j}} + 3.0\,\hat{\mathrm{k}}$.

19. It should be mentioned that an efficient way to work this vector addition problem is with the cosine law for general triangles (and since $\vec{a}$, $\vec{b}$ and $\vec{r}$ form an isosceles triangle, the angles are easy to figure). However, in the interest of reinforcing the usual systematic approach to vector addition, we note that the angle $\vec{b}$ makes with the +x axis is 30°+105° = 135° and apply Eq. 3-5 and Eq. 3-6 where appropriate. (**a**) The x component of $\vec{r}$ is r_x = 10 cos 30° + 10 cos 135° = 1.59 (m). (**b**) The y component of $\vec{r}$ is r_y = 10 sin 30° + 10 sin 135° = 12.1 (m). (**c**) The magnitude of $\vec{r}$ is $(1.59^2+12.1^2)^{1/2}$ = 12.2 (m). (**d**) The angle between $\vec{r}$ and the +x direction is $\tan^{-1}(12.1/1.59)$ = 82.5°.

25. (**a**) As can be seen from Figure 3-30, the point diametriccally opposite the origin (0, 0, 0) has position vector $a\,\hat{\mathrm{i}} + a\,\hat{\mathrm{j}} + a\,\hat{\mathrm{k}}$ and this is the vector along the "body diagonal." (**b**) From the point (a, 0, 0) which corresponds to the position vector $a\,\hat{\mathrm{i}}$, the diametrically opposite point is (0, a, a) with the position vector $a\,\hat{\mathrm{j}} + a\,\hat{\mathrm{k}}$. Thus, the vector along the line is the difference $-a\,\hat{\mathrm{i}} + a\,\hat{\mathrm{j}} + a\,\hat{\mathrm{k}}$. (**c**) If the starting point is (0, a, 0) with the corresponding position vector $a\,\hat{\mathrm{j}}$, the diametrically opposite point is (a, 0, a) with the position vector $a\,\hat{\mathrm{i}} + a\,\hat{\mathrm{k}}$. Thus, the vector along the line is the difference $a\,\hat{\mathrm{i}} - a\,\hat{\mathrm{j}} + a\,\hat{\mathrm{k}}$. (**d**) If the starting point is (a, a, 0) with the corresponding position vector $a\,\hat{\mathrm{i}} + a\,\hat{\mathrm{j}}$, the diametrically opposite point is (0, 0, a) with the position vector $a\,\hat{\mathrm{k}}$. Thus, the vector along the line is the difference $-a\,\hat{\mathrm{i}} - a\,\hat{\mathrm{j}} + a\,\hat{\mathrm{k}}$. (**e**) Consider the vector from the back lower left corner to the front upper right corner. It is $a\,\hat{\mathrm{i}} + a\,\hat{\mathrm{j}} + a\,\hat{\mathrm{k}}$. We may think of it as the sum of the vector $a\,\hat{\mathrm{i}}$ parallel to the x axis and the vector $a\,\hat{\mathrm{j}} + a\,\hat{\mathrm{k}}$ perpendicular to the x axis. The tangent of the angle between the vector and the x axis is the perpendicular component divided by the parallel component. Since the magnitude of the perpendicular component is $\sqrt{a^2 + a^2} = a\sqrt{2}$ and the magnitude of the parallel component is a, $\tan\theta = a\sqrt{2}/a = \sqrt{2}$. Thus $\theta = 54.7°$. The angle between the vector and each of the other two adjacent sides (the y and z axes) is the same as is the angle between any of the other diagonal vectors and any of the cube sides adjacent to them. (**f**) The length of any of the diagonals is given by



$$\sqrt{a^2 + a^2 + a^2} = \sqrt{3}\,a.$$

26. (**a**) With a = 17.0 m and θ = 56.0° we find $a_x = a\cos\theta$ = 9.51 m. (**b**) And $a_y = a\sin\theta$ = 14.1 m. (**c**) The angle relative to the new coordinate system is θ' = (56.0°–18.0°) = 38.0°. Thus, $a_x' = a\cos\theta'$ = 13.4 m. (**d**) And $a_y' = a\sin\theta'$ = 10.5 m.

31. Since $ab\cos\phi = a_x b_x + a_y b_y + a_z b_z$,

$$\cos\phi = \frac{a_x b_x + a_y b_y + a_z b_z}{ab}.$$

The magnitudes of the vectors given in the problem are $a = |\vec{a}| = (3.00^2+3.00^2+3.00^2)^{1/2} = 5.20$,

$$b = |\vec{b}| = (2.00^2+1.00^2+3.00^2)^{1/2} = 3.74.$$

The angle between them is found from

$$\cos\phi = \frac{(3.00)(2.00)+(3.00)(1.00)+(3.00)(3.00)}{(5.20)(3.74)} = 0.926.$$

The angle is ϕ = 22°.

34. Using that $\hat{\mathrm{i}}\times\hat{\mathrm{j}} = \hat{\mathrm{k}}$, $\hat{\mathrm{j}}\times\hat{\mathrm{k}} = \hat{\mathrm{i}}$, and $\hat{\mathrm{k}}\times\hat{\mathrm{i}} = \hat{\mathrm{j}}$, we obtain

$$2\vec{A}\times\vec{B} = 2(2.00\,\hat{\mathrm{i}} + 3.00\,\hat{\mathrm{j}} - 4.00\,\hat{\mathrm{k}})$$

$$\times(-3.00\,\hat{\mathrm{i}} + 4.00\,\hat{\mathrm{j}} + 2.00\,\hat{\mathrm{k}}) = (44.0\,\hat{\mathrm{i}} + 16.0\,\hat{\mathrm{j}} + 34.0\,\hat{\mathrm{k}}).$$

Next, making use of $\hat{\mathrm{i}}\cdot\hat{\mathrm{i}} = \hat{\mathrm{j}}\cdot\hat{\mathrm{j}} = \hat{\mathrm{k}}\cdot\hat{\mathrm{k}} = 1$, $\hat{\mathrm{i}}\cdot\hat{\mathrm{j}} = \hat{\mathrm{j}}\cdot\hat{\mathrm{k}} = \hat{\mathrm{k}}\cdot\hat{\mathrm{i}} = 0$, we obtain

$$3\vec{C}\cdot(2\vec{A}\times\vec{B})$$

$$= (7.00\,\hat{\mathrm{i}} - 8.00\,\hat{\mathrm{j}})\cdot(44.0\,\hat{\mathrm{i}} + 16.0\,\hat{\mathrm{j}} + 34.0\,\hat{\mathrm{k}})$$

$$= 3[(7.00)(44.0) - (8.00)(16.0)] = 540.$$

39AP. The point P is displaced vertically by $2R$, where R is the radius of the wheel. It is displaced horizontally by half the circumference of the wheel, or πR. Since R = 0.450 m, the horizontal component of the displacement is 1.414 m and the vertical component of the displacement is 0.900 m. If the x axis is horizontal and the y axis is vertical, the vector displacement (in meters) is $\vec{r}$ = 1.414 m $\hat{\mathrm{i}}$ +0.900 m $\hat{\mathrm{j}}$. The displacement has a magnitude of

$$|\vec{r}| = \sqrt{(\pi R)^2 + (2R)^2} = R\sqrt{\pi^2 + 4} = 1.68\ \text{(m)},$$

and an angle of

$$\tan^{-1}\left(\frac{2R}{\pi R}\right) = \tan^{-1}\left(\frac{2}{\pi}\right) = 32.5°$$

above the floor. In physics there are no "exact" measurements, yet that angle computation seemed to yield something *exact*. However, there has to be some uncertainty in the observation that the wheel rolled half of a revolution, which introduces some indefiniteness in our result.

55AP. The two vectors are given by

$$\vec{A} = 8.00(\cos 130°\,\hat{\mathrm{i}} + \sin 130°\,\hat{\mathrm{j}}) = -5.14\,\hat{\mathrm{i}} + 6.13\,\hat{\mathrm{j}}$$

and $\vec{B} = B_x\,\hat{\mathrm{i}} + B_y\,\hat{\mathrm{j}} = -7.72\,\hat{\mathrm{i}} - 9.20\,\hat{\mathrm{j}}$.

(**a**) The dot product of $5\vec{A}\cdot\vec{B}$ is

$$5\vec{A}\cdot\vec{B} = 5(-5.14\,\hat{\mathrm{i}} + 6.13\,\hat{\mathrm{j}})\cdot(-7.72\,\hat{\mathrm{i}} - 9.20\,\hat{\mathrm{j}})$$

$$= 5[(-5.14)(-7.72) + (6.13)(-9.20)] = -83.4.$$

(**b**) In unit vector notation

$$4\vec{A}\times 3\vec{B} = 12\,\vec{A}\times\vec{B} = 12(-5.14\,\hat{\mathrm{i}} + 6.13\,\hat{\mathrm{j}})\times$$

$$(-7.72\,\hat{\mathrm{i}} - 9.20\,\hat{\mathrm{j}}) = 12(94.6\,\hat{\mathrm{k}}) = 1.14\times10^3\,\hat{\mathrm{k}}.$$

(**c**) We note that the azimuthal angle is undefined for a vector along the z axis. Thus, our result is "1.14×10^3, θ not defined, and ϕ = 0°." (**d**) Since $\vec{A}$ is in the xy plane, and $\vec{A}\times\vec{B}$ is perpendicular to that plane, then the answer is 90°. (**e**) Clearly, $\vec{A} + 3.00\,\hat{\mathrm{k}} = -5.14\hat{\mathrm{i}} + 6.13\hat{\mathrm{j}} + 3.00\,\hat{\mathrm{k}}$. (**f**) The Pythagorean theorem yields magnitude $A = (5.14^2+6.13^2+3.00^2)^{1/2} = 8.54$. The azimuthal angle is $\theta = 130°$, just as it was in the problem statement ($\vec{A}$ is the projection onto to the *xy* plane of the new vector created in part (e)). The angle measured from the +z axis is $\phi = \cos^{-1}(3.00/8.54) = 69.4°$.

77AP.* The area of a triangle is half the product of its base and altitude. The base is the side formed by vector $\vec{a}$. Then the altitude is $b\sin\phi$ and the area is

$$A = \frac{1}{2}ab\sin\phi = \frac{1}{2}|\vec{a}\times\vec{b}|.$$

為由 $\vec{A}$ 及 $\vec{B}$ 相鄰邊組成之平行四邊形的面積。

(如發現錯誤煩請告知 jyang@mail.ntou.edu.tw, Thanks.)

***Ex. Pb.*3-65**

- “**新一代 GPS**”，艾胥利，科學人 2003 年 10 月號。
- “**GPS：讓路痴不再迷路**”，哈奇森，科學人 2004 年 6 月號。
- “**GPS：讓飛航更安全、更準時**”，翁千婷，科學人 2004 年 6 月號。

第 3 章 向量

Unit vectors: $\hat{i} \parallel x$ axis, $\hat{j} \parallel y$ axis, $\hat{k} \parallel z$ axis,

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$, $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0$;

$A = |\vec{A}| = (A_x^2+A_y^2+A_z^2)^{1/2}$,

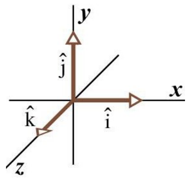
$\vec{A}\cdot\vec{B} \equiv AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$;

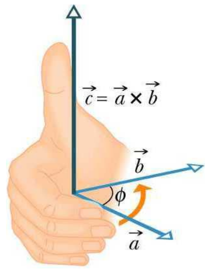
$\vec{A}\times\vec{B} = C\hat{c}$, $C \equiv AB\sin\theta > 0$, where $0 \le \theta \le 180°$,

and $\vec{C}\perp\vec{A}$ & $\vec{C}\perp\vec{B}$.

$\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$;

$\vec{A}\times\vec{B} = (A_yB_z-A_zB_y)\,\hat{i} + (A_zB_x-A_xB_z)\,\hat{j} + (A_xB_y-A_yB_x)\,\hat{k}$





- 備忘錄